Meta-centralizers of non locally compact group algebras.

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Abstract

Meta-centralizers of non-locally compact group algebras are studied. Theorems about their representations with the help of families of generalized measures are proved. Isomorphisms of group algebras are investigated in relation with meta-centralizers.

[1]

1 Introduction.

Locally compact group algebras are rather well investigated and play very important role in mathematics [10, 12, 24, 14, 16]. Left centralizers of locally compact group algebras were studied in [27]. In all those works Haar measures on locally compact groups were used. Haar measures are invariant or quasi-invariant relative to left or right shifts of the entire locally compact group [6, 10, 12, 24]. According to the A.Weil theorem if a topological group has a non-trivial borelian measure quasi-invariant relative to left or right shifts of the entire group, then it is locally compact.

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On the other hand, the theory of non locally compact groups and their representations differ drastically from that of the locally compact case (see [2, 3, 11, 18, 19, 21] and references therein). Measures on non locally compact groups quasi-invariant relative to proper dense subgroups were constructed in [4, 7, 8, 18, 19, 20, 21, 23].

This article continues investigations of non locally compact group algebras [17, 19, 22]. The present paper is devoted to centralizers of non-locally compact group algebras, which are substantially different from that of locally compact groups. Their definition in the non locally compact groups setting is rather specific and they are already called meta-centralizers. Theorems about their representations with the help of families of generalized measures are proved. Isomorphisms of group algebras are investigated in relation with meta-centralizers. The main results of this paper are Theorems 8-10 and 14. They are obtained for the first time.

Henceforth definitions and notations of [17] are used.

2 Group algebra

To avoid misunderstandings we first present our definitions and notations.

1. Definition. Let Λ be a directed set and let $\{G_\alpha : \alpha \in \Lambda\}$ be a family of topological groups with completely regular (i.e. $T_1 \cap T_{3\frac{1}{2}}$) topologies τ_α satisfying the following restrictions:

(1) $\theta^\beta_\alpha : G_\beta \to G_\alpha$ is a continuous algebraic embedding, $\theta^\beta_\alpha(G_\beta)$ is a proper subgroup in G_α for each $\alpha < \beta \in \Lambda$;

(2) $\tau_\alpha \cap \theta^\beta_\alpha(G_\beta) \subset \theta^\beta_\alpha(\tau_\beta)$ and $\theta^\beta_\alpha(G_\beta)$ is dense in G_α for each $\alpha < \beta \in \Lambda$; then $(\theta^\beta_\alpha)^{-1} : \theta^\beta_\alpha(G_\beta, \tau_\beta) \to (G_\beta, \tau_\beta)$ is considered as the continuous homomorphism;

(3) G_α is complete relative to the left uniformity with entourages of the diagonal of the form $\mathcal{U} = \{(h, g) : h, g \in G_\alpha; h^{-1}g \in U\}$ with neighborhoods U of the unit element e_α in G_α, $U \in \tau_\alpha$, $e_\alpha \in U$;

(4) for each $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$ the embedding $\theta^\beta_\alpha : (G_\beta, \tau_\beta) \hookrightarrow (G_\alpha, \tau_\alpha)$ is precompact, that is by our definition for every open set U in G_β containing the unit element e_β a neighborhood $V \in \tau_\beta$ of e_β exists so that $V \subset U$ and $\theta^\beta_\alpha(V)$ is precompact in G_α, i.e. its closure $cl(\theta^\beta_\alpha(V))$ in G_α is compact, where $\phi : \Lambda \to \Lambda$ is an increasing marked mapping.

2. Conditions. Henceforward it is supposed that Conditions $(1-5)$ are satisfied:

(1) $\mu_\alpha : \mathcal{B}(G_\alpha) \to [0,1]$ is a probability measure on the Borel σ-algebra $\mathcal{B}(G_\alpha)$ of a group G_α from §1 with $\mu_\alpha(G_\alpha) = 1$ so that

(2) μ_α is quasi-invariant relative to the right and left shifts on $h \in \theta^\beta_\alpha(G_\beta)$ for each $\alpha < \beta \in \Lambda$, where $\rho^r_{\mu_\alpha}(h,g) = (\mu^h_\alpha)(dg)/\mu(dg)$ and $\rho^l_{\mu_\alpha}(h,g) = (\mu_{\alpha,h})(dg)/\mu(dg)$ denote quasi-invariance μ_α-integrable factors, $\mu^h_\alpha(S) = \mu(Sh^{-1})$ and $\mu_{\alpha,h}(S) = \mu_\alpha(h^{-1}S)$ for each Borel subset S in G_α;

(3) a density $\psi_\alpha(g) = \mu_\alpha(dg^{-1})/\mu_\alpha(dg)$ relative to the inversion exists and it is μ_α-integrable;

(4) a subset $W_\alpha \in \mathcal{A}(G_\alpha)$ exists such that $\rho^r_{\mu_\alpha}(h,g)$ and $\rho^l_{\mu_\alpha}(h,g)$ are continuous on $\theta^\beta_\alpha(G_\beta) \times W_\alpha$ and $\psi_\alpha(g)$ is continuous on W_α with $\mu_\alpha(W_\alpha) = 1$ for each $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$;

(5) each measure μ_α is Borel regular and radonian,
where the completion of $\mathcal{B}(G_\alpha)$ by all μ_α-zero sets is denoted by $\mathcal{A}(G_\alpha)$.

3. Notation. Denote by $L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ a subspace in $L^1(G_\alpha, \mu_\alpha, \mathbf{F})$, which is the completion of the linear space $L^0(G_\alpha, \mathbf{F})$ of all (μ_α-measurable) simple functions

$$f(x) = \sum_{j=1}^n b_j \chi_{B_j}(x),$$

where $b_j \in \mathbf{F}$, $B_j \in \mathcal{A}(G_\alpha)$, $B_j \cap B_k = \emptyset$ for each $j \ne k$, χ_B denotes the characteristic function of a subset B, $\chi_B(x) = 1$ for each $x \in B$ and $\chi_B(x) = 0$ for every $x \in G_\alpha \setminus B$, $n \in \mathbf{N}$, where $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$. A norm on $L^1_{G_\beta}(G_\alpha)$ is by our definition given by the formula:

$$(1) \quad \|f\|_{L^1_{G_\beta}(G_\alpha)} := \sup_{h \in \theta^\beta_\alpha(G_\beta)} \|f_h\|_{L^1(G_\alpha)} < \infty,$$

where $f_h(g) := f(h^{-1}g)$ for $h, g \in G_\alpha$, $L^1(G_\alpha, \mu_\alpha, \mathbf{F})$ is the usual Banach space of all μ_α-measurable functions $u : G_\alpha \to \mathbf{F}$ such that

$$(2) \quad \|u\|_{L^1(G_\alpha)} = \int_{G_\alpha} |u(g)| \mu_\alpha(dg) < \infty.$$

Suppose that

(3) $\phi : \Lambda \to \Lambda$ is an increasing mapping, $\alpha < \phi(\alpha)$ for each $\alpha \in \Lambda$. We consider the space

(4) $L^\infty(L^1_{G_\beta}(G_\alpha,\mu_\alpha,\mathbf{F}) : \alpha < \beta \in \Lambda) := \{f = (f_\alpha : \alpha \in \Lambda);\ f_\alpha \in L^1_{G_\beta}(G_\alpha,\mu_\alpha,\mathbf{F})$ for each $\alpha \in \Lambda$; $\|f\|_\infty := \sup_{\alpha\in\Lambda} \|f_\alpha\|_{L^1_{G_\beta}(G_\alpha)} < \infty$, where $\beta = \phi(\alpha)\}$.

When measures μ_α are specified, spaces are denoted shortly by $L^1_{G_\beta}(G_\alpha,\mathbf{F})$ and $L^\infty(L^1_{G_\beta}(G_\alpha,\mathbf{F}) : \alpha < \beta \in \Lambda)$ respectively.

4. Definition. Let the algebra $\mathcal{E} := L^\infty(L^1_{G_\beta}(G_\alpha,\mu_\alpha,\mathbf{F}) : \alpha < \beta \in \Lambda)$ be supplied with the multiplication $f\tilde{\star}u = w$ such that

$$(1) \quad w_\alpha(g) = (f_\beta\tilde{*}u_\alpha)(g) = \int_{G_\beta} f_\beta(h)u_\alpha(\theta^\beta_\alpha(h)g)\mu_\beta(dh)$$

for every $f,u \in \mathcal{E}$ and $g \in G = \prod_{\alpha\in\Lambda} G_\alpha$, where $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$, $\beta = \phi(\alpha)$, $\alpha \in \Lambda$.

If a bounded linear transformation $T : \mathcal{E} \to \mathcal{E}$ satisfies Conditions $(2,3)$:

(2) $Tf = (T_\alpha f_\alpha : \alpha \in \Lambda)$, $T_\alpha : L^1_{G_\beta}(G_\alpha,\mu_\alpha,\mathbf{F}) \to L^1_{G_\beta}(G_\alpha,\mu_\alpha,\mathbf{F})$ for each $\alpha \in \Lambda$,

(3) $T(f\tilde{\star}u) = f\tilde{\star}(Tu)$

for each $f,u \in \mathcal{E}$, then T is called a left meta-centralizer.

5. Definitions. Let X be a topological space, let $C(X,\mathbf{R})$ be the space of all continuous functions $f : X \to \mathbf{R}$, while $C_b(X,\mathbf{R})$ be the space of all bounded continuous functions with the norm

(1) $\|f\| := \sup_{x\in X} |f(x)| < \infty$.

Suppose that $\mathcal{F}$ is the least σ-algebra on X containing the algebra $\mathcal{Z}$ of all functionally closed subsets $A = f^{-1}(0)$, $f \in C_b(X,\mathbf{R})$. A finitely additive non-negative mapping $m : \mathcal{F} \to [0,\infty)$ such that

(2) $m(A) = \sup\{m(B) : B \in \mathcal{Z}, B \subset A\}$

for each $A \in \mathcal{F}$ is called (a finitely additive) measure. A generalized measure is the difference of two measures. Denote by $M(X) = M(X,\mathbf{R})$ the family of all generalized (finitely additive) measures.

For short "generalized" may be omitted, when m is considered with values in $\mathbf{R}$.

6. Theorem (A.D. Alexandroff [26]). *$M(X)$ is the topologically dual space to $C_b(X,\mathbf{R})$, that is for each bounded linear functional J on $C_b(X,\mathbf{R})$ there exists a unique generalized (finitely additive) measure $m \in M(X)$ such that*

(1) *$J(f) = \int_X f dm$ for each $f \in C_b(X,\mathbf{R})$,*

each measure $m \in M(X)$ defines a unique continuous linear functional by Formula (1)*. Moreover,*

(2) $\|J\| = \|m\|$.

7. Definitions. A bounded linear functional J on $C_b(X, \mathbf{R})$ is called σ-smooth, if

(1) $\lim_n J(f_n) = 0$

for each sequence f_n in $C_b(X, \mathbf{R})$ such that $0 \le f_{n+1}(x) \le f_n(x)$ and $\lim_n f_n(x) = 0$ for each point $x \in X$. The linear space of all σ-smooth linear functionals is denoted by $M_\sigma(X) = M_\sigma(X, \mathbf{R})$.

A bounded linear functional J on $C_b(X, \mathbf{R})$ is called tight, if Formula (1) is fulfilled for each net f_α in $C_b(X, \mathbf{R})$ such that $\|f_\alpha\| \le 1$ for each α and f_α tends to zero uniformly on each compact subset K in X. The space of all tight linear functionals is denoted by $M_t(X) = M_t(X, \mathbf{R})$.

If $m_1, m_2 \in M(X)$, then $m = m_1 + im_2$ is a complex-valued measure, their corresponding spaces are denoted by $M(X, \mathbf{C})$, $M_\sigma(X, \mathbf{C}) = M_\sigma(X) + iM_\sigma(X)$ and $M_t(X, \mathbf{C}) = M_t(X) + iM_t(X)$.

8. Theorem. *Let $\mathcal{E}$ be a real $\mathbf{F} = \mathbf{R}$ or complex $\mathbf{F} = \mathbf{C}$ algebra (see §4), let also T be a left meta-centralizer on $\mathcal{E}$. Then there exists a family $\nu = (\nu_\alpha : \alpha \in \Lambda)$ of generalized $\mathbf{F}$-valued measures ν_α on G_α of bounded variation such that*

(1) $Tf = \nu \tilde{\star} f$*, where*

$$(2) \quad (T_\alpha f_\alpha)(g) = (\nu_\beta \tilde{*} f_\alpha)(g) = \int_{G_\beta} \nu_\beta(dh) f_\alpha(\theta^\beta_\alpha(h)g)$$

for each $\alpha \in \Lambda$ and $g \in G_\alpha$ with $\beta = \phi(\alpha)$.

Proof. For each $\beta \in \Lambda$ and a neutral element $e_\beta \in G_\beta$ we consider a basis of its neighborhoods $\{V_{a,\beta} : a \in \Psi_\beta\}$ such that $cl_{G_\alpha} \theta^\beta_\alpha(V_{a,\beta})$ is compact in (G_α, τ_α), where Ψ_β is a set, $cl_X A$ denotes the closure of a set A in a topological space X. The set Ψ_β is directed by the inclusion: $a \le b \in \Psi_\beta$ if and only if $V_{b,\beta} \subseteq V_{a,\beta}$.

There is a natural continuous linear restriction mapping $p^U_V : C_b(U, \mathbf{F}) \to C_b(V, \mathbf{F})$ for each closed subsets U and V in G_β such that $V \subset U$, where $p^U_V(f) = f|_V$ for each $f \in C_b(U, \mathbf{F})$. At the same time, if U is compact, then each continuous bounded function g on V with values in $\mathbf{F}$ has a continuous extension $\pi^V_U(g)$ on U with values in $\mathbf{F}$ such that

$$\|g\|_{C_b(V,\mathbf{F})} \le \|\pi^V_U(g)\|_{C_b(U,\mathbf{F})} \le 2\|g\|_{C_b(V,\mathbf{F})}$$

due to Tietze-Uryson Theorem 2.1.8 [9], since G_β is T_0 and hence completely regular by Theorem 8.4 [12] and each Huasdorff compact space is normal by Theorems 5.1.1 and 5.1.5 [9]. Thus there exists a linear continuous embedding $\pi^V_U : C_b(V, \mathbf{F}) \hookrightarrow C_b(U, \mathbf{F})$.

The probability measure μ_β on G_β is Borel regular and radonian hence there exists a σ-compact subset X_β in G_β such that $\mu_\beta(X_\beta) = 1$, i.e. X_β is the countable union of compact subsets $X_{\beta,n}$ in (G_β, τ_β) with $X_{\beta,n} \subset X_{\beta,n+1}$ for each natural number n.

We put

(3) $q_{a,\beta} := \chi_{V_{a,\beta}}/\mu_\beta(V_{a,\beta})$,

where χ_A denotes the characteristic function of a subset A in G_β, $\chi_A(x) = 1$ for each $x \in A$ while $\chi_A(x) = 0$ for each $x \notin A$. In view of Proposition 17.7 [19] (see also Lemma 13 [17]) the net $\{q_{a,\beta} : a \in \Psi_\beta\}$ is an approximation of the identity relative to the convolution:

(4) $\lim_a q_{a,\beta} \tilde{*} f_\alpha = f_\alpha$

for each $f_\alpha \in L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$. From Formulas $(2, 4)$ and $4(1-3)$ it follows that

$$(5) \quad T_\alpha f_\alpha = T_\alpha[\lim_a q_{a,\beta} \tilde{*} f_\alpha] = \lim_a q_{a,\beta} \tilde{*} [T_\alpha f_\alpha].$$

Then $q_{a,\beta} \tilde{*} [T_\alpha \cdot] : L^1_{G_\beta}(G_\alpha) \to L^1_{G_\beta}(G_\alpha)$ is a continuous linear operator for each $a \in \Psi_\beta$ and $\alpha \in \Lambda$, particularly, for each f_α in the space $C_b(G_\alpha, \mathbf{F})$ of all bounded continuous functions on G_α with values in the field $\mathbf{F}$, where

(6) $\|f_\alpha\|_{C_b} := \sup_{x \in G_\alpha} |f_\alpha(x)| < \infty$

for each $f_\alpha \in C_b(G_\alpha, \mathbf{F})$. The restriction of each $f_\alpha \in C_b(G_\alpha, \mathbf{F})$ on $\theta^\beta_\alpha(G_\beta)$ is bounded and continuous, while $C_b(G_\beta, \mathbf{F})$ is dense in $L^1_{G_\gamma}(G_\beta, \mu_\beta, \mathbf{F})$ with $\gamma = \phi(\beta)$ (see also Lemma 17.8 and Proposition 17.9 [19]).

This implies that an adjoint operator $B = T^*$ exists relative to the $\tilde{*}$ multiplication according to the formula:

$$(7) \quad (v_\beta \tilde{*} [T_\alpha \bar{f}_\alpha])(x) = \int_{G_\beta} v_\beta(h)[T_\alpha \bar{f}_\alpha](\theta^\beta_\alpha(h)x)\mu_\beta(dh)$$

$$=: \int_{G_\beta} (B_\beta v_\beta)(h)\bar{f}_\alpha(\theta^\beta_\alpha(h)x)\mu_\beta(dh)$$

for each $v, f \in \mathcal{E}$, where $x \in G_\alpha$, $\bar{z}$ denotes the complex conjugated number of $z \in \mathbf{C}$. The operator B_β is bounded and linear from $L^1_{G_\gamma}(G_\beta)$ into itself,

since from Formula (7) the estimate follows:

$$(8)\quad \|B_\beta\| \le \sup_{s\in\theta^\gamma_\beta(G_\gamma),\ t\in\theta^\beta_\alpha(G_\beta),\ 0\ne v_\beta\in L^1_{G_\gamma}(G_\beta),\ 0\ne f_\alpha\in L^1_{G_\beta}(G_\alpha)}$$

$$\frac{|\int_{G_\alpha}\int_{G_\beta} v_\beta(sh)[T_\alpha\bar{f}_\alpha](\theta^\beta_\alpha(h)tx)\mu_\beta(dh)\mu_\alpha(dx)|}{\|v_\beta\|_{L^1_{G_\gamma}(G_\beta)}\|f_\alpha\|_{L^1_{G_\beta}(G_\alpha)}} \le \|T_\alpha\| < \infty.$$

The family of bounded linear operators $\{(B_\beta q_{a,\beta})\tilde{*} : \ a \in \Psi_\beta\}$ from $L^1_{G_\beta}(G_\alpha)$ into $L^1_{G_\beta}(G_\alpha)$ is pointwise bounded and hence by the Banach-Steinhaus Theorem (11.6.1) [25] it is uniformly bounded:

$$(B1)\quad \sup_{a\in\Psi_\beta}\|(B_\beta q_{a,\beta})\tilde{*}\| < \infty.$$

Therefore inequality (8) leads to the conclusion that $B_\beta q_{a,\beta} =: h_{a,\beta} \in L^1_{G_\gamma}(G_\beta,\mu_\beta,\mathbf{F})$ for every $a\in\Psi_\beta$ and $\beta\in\Lambda$. Each function $h_{a,\beta}$ induces the linear functional

$$(9)\quad F_{a,\beta}(g_\beta) := \int_{G_\beta} g_\beta(x)\bar{h}_{a,\beta}(x)\mu_\beta(dx).$$

Without loss of generality we choose $V_{a,\beta}$ such that $cl_{G_\alpha}V_{a,\beta}$ is compact in (G_α,τ_α) for each $a\in\Psi_\beta$. Certainly, if $f\in L^1_{G_\gamma}(G_\beta,\mu_\beta,\mathbf{F})$, then $f\in L^1(G_\beta,\mu_\beta,\mathbf{F})$ and

$$(10)\quad \|f\|_{L^1(G_\beta,\mu_\beta,\mathbf{F})} \le \|f\|_{L^1_{G_\gamma}(G_\beta,\mu_\beta,\mathbf{F})} < \infty.$$

There is the embedding $C_b(G_\beta,\mathbf{F})\subset L^1_{G_\gamma}(G_\beta,\mu_\beta,\mathbf{F})$ and

$$(11)\quad \|f\|_{L^1_{G_\gamma}(G_\beta,\mu_\beta,\mathbf{F})} \le \|f\|_{C_b(G_\beta,\mathbf{F})} < \infty$$

for each $f\in C_b(G_\beta,\mathbf{F})$, since μ_β is the probability measure on G_β.

If $f\in L^1_{G_\gamma}(G_\beta)$, then $s\mapsto f\tilde{*}s$ is a continuous linear operator from $C_b(G_\beta,\mathbf{F})$ into $C_b(G_\beta,\mathbf{F})$. This follows from the formulas:

$$(12)\quad (f\tilde{*}s)(g) = \int_{G_\beta} f(h)s(hg)\mu_\beta(dh),$$

where $g\in G_\beta$ and

$$\sup_g |(f\tilde{*}s)(g)| \le \|s\|_{C_b}\int_{G_\beta}|f(h)|\mu_\beta(dh) \le \|s\|_{C_b}\|f\|_{L^1(G_\beta)} \le \|s\|_{C_b}\|f\|_{L^1_{G_\gamma}(G_\beta)}.$$

It remains to verify that the function $(f\tilde{*}s)(g)$ is continuous for each f and s as just above. For the proof consider the term

$$(13) \quad |(f\tilde{*}s)(g_1) - (f\tilde{*}s)(g_2)| = |\int_{G_\beta} f(h)[s(hg_1) - s(hg_2)]\mu_\beta(dh)|.$$

From $f \in L^1_{G_\gamma}(G_\beta)$ and $s \in C_b(G_\beta, \mathbf{F})$ it follows that for each $\epsilon > 0$ there exists a compact subset V in G_β such that $\int_{G_\beta \setminus V} |f(h)|\mu_\beta(dh) < \epsilon$ and hence $\int_{G_\beta \setminus V} |f(h)[s(hg_1) - s(hg_2)]|\mu_\beta(dh) < \delta$, where $0 < \delta = \epsilon 2\|s\|_{C_b}$. Indeed, for each $\delta > 0$ there exists a simple function $q \in L^1_{G_\gamma}(G_\beta)$ such that $\|f - q\|_{L^1_{G_\gamma}(G_\beta)} < \delta$ and hence the measure $|f(h)|\mu_\beta(dh)$ is radonian together with $|q(h)|\mu_\beta(dh)$. At the same time, certainly, $\int_V |f(h)|\mu_\beta(dh) \le \|f\|_{L^1(G_\beta)}$.

On the other hand, $[s(hg_1) - s(hg_2)]$ is uniformly continuous on V by the variable h, since V is compact and s is the continuous function. For each symmetric open neighborhood $U = U^{-1}$ of the neutral element e_β in G_β there exists a finite family of elements $p_1, ..., p_n \in G_\beta$ such that $V \subset p_1U \cup ... \cup p_nU$, since V is compact. Thus $VU \subset p_1U^2 \cup ... \cup p_nU^2$. Consider a family of symmetric open neighborhoods $U_k = U_k^{-1}$ of e_β such that $\{p_kU_k : k \in \omega\}$ is a covering of V and $|s(hg_1) - s(hg_2)| < \epsilon$ for each $h \in p_kU_k$ and $g_1, g_2 \in U_k$, where $p_k \in G_\beta$ for each k, whilst ω is an ordinal. The covering p_kU_k of V has a finite subcovering for $k \in M$, where M is a finite subset in ω. Thus for each $\epsilon > 0$ there exists a symmetric neighborhood $U \subseteq \bigcap_{k \in M} U_k$ of e_β such that $|s(hg_1) - s(hg_2)| < \epsilon$ for each $h \in V$ and $g_1, g_2 \in U$. Therefore,

$$|(f\tilde{*}s)(g_1) - (f\tilde{*}s)(g_2)| \le \delta + \epsilon\|f\|_{L^1} = \epsilon(\|f\|_{L^1} + 2\|s\|_{C_b})$$

for each $g_1, g_2 \in U$. Thus

(14) $f\tilde{*}s \in C_b(G_\beta, \mathbf{F})$ for each $f \in L^1_{G_\gamma}(G_\beta, \mathbf{F})$ and $s \in C_b(G_\beta, \mathbf{F})$.

This implies that

(15) $C_b(G_\beta, \mathbf{F}) \ni s \mapsto (f\tilde{*}s)(e_\beta) \in \mathbf{F}$

is the continuous linear functional on $C_b(G_\beta, \mathbf{F})$. In particular each operator $(B_\beta q_{a,\beta})\tilde{*}$ indices the continuous linear functional

(16) $J_{a,\beta}(s) = [(B_\beta q_{a,\beta})\tilde{*}s](e_\beta)$ on $C_b(G_\beta, \mathbf{F})$.

There are the inclusions $M_t(X) \subset M_\sigma(X) \subset M(X)$ (see §I.4 [26] and Definitions 5, 7 and Theorem 6 above) and for $X = G_\beta$ in particular. On the other hand, each $w_{a,\beta}(dx) := (B_\beta q_{a,\beta})(x)\mu_\beta(dx)$ is the radonian measure on G_β, i.e. belongs to the space $M_t(G_\beta, \mathbf{F})$ of radonian measures on G_β.

Let Φ_β be a family of all left-invariant pseudo-metrics on (G_β, τ_β) providing its left uniformity denoted by $\mathcal{L}_\beta$ (see §8.1.7 [9] and Condition 1(3)). This means that each $\kappa \in \Phi_\beta$ satisfies the restrictions:

$(P1)$ $\kappa(x,y) \geq 0$,

$(P2)$ $\kappa(x,x) = 0$,

$(P3)$ $\kappa(x,y) = \kappa(y,x)$,

$(P4)$ $\kappa(x,y) \leq \kappa(x,z) + \kappa(z,y)$

$(P5)$ $\kappa(zx,zy) = \kappa(x,y)$ for each $x,y,z \in G_\beta$.

The family Φ_β is directed: $\kappa_1 \leq \kappa \in \Phi_\beta$ if and only if $\kappa_1(x,y) \leq \kappa(x,y)$ for each $x,y \in G_\beta$; without loss of generality for each $\kappa, \kappa_1 \in \Phi_\beta$ there exists $\kappa_2 \in \Phi_\beta$ such that $\kappa \leq \kappa_2$ and $\kappa_1 \leq \kappa_2$, since $\kappa + \kappa_1 \in \Phi_\beta$. Each pseudo-metric $\kappa \in \Phi_\beta$ defines the equivalence relation: $x\Xi_\kappa y$ if and only if $\kappa(x,y) = 0$. Then as the uniform space $(G_\beta, \mathcal{L}_\beta)$ has the projective limit decomposition (i.e. the limit of the inverse mapping system)

$$G_\beta = \lim\{G_{\beta,\kappa}, \pi^\kappa_\omega, \Phi_\beta\},$$

where $G_{\beta,\kappa} := G_\beta/\Xi_\kappa$ denotes the quotient uniform space with the quotient uniformly, π_κ is a uniformly continuous mapping from G_β onto $G_{\beta,\kappa}$, π^κ_ω are uniformly continuous mappings from $G_{\beta,\kappa}$ onto $G_{\beta,\omega}$ for each $\omega \leq \kappa \in \Psi_\beta$ such that $\pi^\omega_\xi \circ \pi^\kappa_\omega = \pi^\kappa_\xi$ and $\pi_\omega = \pi^\kappa_\omega \circ \pi_\kappa$ for each $\xi \leq \omega \leq \kappa \in \Phi_\beta$ (see §§8.2.B, 2.5.F and Proposition 2.4.2 [9] or [13]). Moreover, the equality is satisfied: $\{y \in G_\beta : \ x\Xi_\kappa y\} = x\Omega_{\beta,\kappa}$ with $\Omega_{\beta,\kappa} := \{y \in G_\beta : \ e_\beta \Xi_\kappa y\}$, since $\kappa(x,y) = 0$ if and only if $\kappa(e_\beta, x^{-1}y) = 0$ by Property $(P5)$, where e_β denotes the neutral element in the group G_β. That is, $G_{\beta,\kappa}$ is called the homogeneous quotient uniform space.

At the same time the σ-compact subset X_β is dense in G_β, since $\mu_\beta(U) > 0$ for each open subset U in G_β, but $\mu_\beta(X_\beta) = \mu_\beta(G_\beta) = 1$ (see the proof above). Therefore, $\pi_\kappa(X_\beta)$ is dense in $G_{\beta,\kappa}$. Then $\pi_\kappa(X_{\beta,n})$ is compact for each $\kappa \in \Phi_\beta$ as the continuous image of the compact space according to Theorem 3.1.10 [9], consequently, $\pi_\kappa(X_\beta) = \bigcup_{n=1}^\infty \pi_\kappa(X_{\beta,n})$ is σ-compact. On the other hand, $G_{\beta,\kappa}$ is metrizable and complete, since $(G_\beta, \mathcal{L}_\beta)$ is complete. Therefore, the topological space $\pi_\kappa(X_\beta)$ is separable, since each $\pi_\kappa(X_{\beta,n})$ is separable by Theorems 4.3.5 and 4.3.27 [9] and $\pi_\kappa(X_\beta) = \bigcup_{n=1}^\infty \pi_\kappa(X_{\beta,n})$. This implies that each metrizable space $G_{\beta,\kappa}$ is separable and complete.

The spaces $C_b(G_\beta, \mathbf{F})$ and $C_b^*(G_\beta, \mathbf{F})$ form the dual pair (see §§9.1 and 9.2 [25]). Then we get that the space of bounded continuous functions $C_b(G_\beta, \mathbf{F})$

has the inductive limit representation $C_b(G_\beta,\mathbf{F}) = ind-\lim_{\Phi_\beta} C_b(G_{\beta,\kappa},\mathbf{F})$, while its topologically dual space has the projective limit decomposition $C_b^*(G_\beta,\mathbf{F}) = pr-\lim_{\Phi_\beta} C_b^*(G_{\beta,\kappa},\mathbf{F})$ (see §§9.4, 9.9, 12.2, 12.202 [25] and also the note after Theorem 2.5.14 in [9]). This implies that $\nu_\beta \in M(G_\beta,\mathbf{F})$ if and only if

$$(M1)\quad \nu_\beta = \lim\{\nu_{\beta,\kappa},\pi^\kappa_\omega,\Phi_\beta\},$$

where $\nu_{\beta,\kappa}\in M(G_{\beta,\kappa},\mathbf{F})$ for each $\kappa\in\Phi_\beta$ so that

$$(M2)\quad \nu_\beta(\pi_\omega^{-1}(C)) = \nu_{\beta,\omega}(C) \text{ and } \nu_{\beta,\kappa}((\pi^\kappa_\omega)^{-1}(C)) = \nu_{\beta,\omega}(C)$$

for every $C\in\mathcal{B}(G_{\beta,\omega})$ and $\omega\le\kappa\in\Phi_\beta$.

Then we consider the measure net $\{w_{a,\beta,\kappa}: a\in\Psi_\beta\}$ for each $\kappa\in\Phi_\beta$ corresponding to measures $w_{a,\beta}(dx) = (B_\beta q_{a,\beta})(x)\mu_\beta(dx)$ according to Formula $(M2)$, where $x\in G_\beta$. Since the measure $w_{a,\beta}(dx)$ is absolutely continuous relative to the radonian measure μ_β, then $w_{a,\beta}$ is also radonian. Therefore, there is the inclusion $\{w_{a,\beta,\kappa}: a\in\Psi_\beta\}\subset M_t(G_{\beta,\kappa},\mathbf{F})$ and it is known that $M_t(Y,\mathbf{F})\subset M_\sigma(Y,\mathbf{F})\subset M(Y,\mathbf{F})$ for a completely regular topological space Y. Thus the measure net $\{w_{a,\beta}: a\in\Psi_\beta\}$ weakly converges to some measure ν_β in $M(G_\beta,\mathbf{F})$ if and only if the net $\{w_{a,\beta,\kappa}: a\in\Psi_\beta\}$ weakly converges in $M(G_{\beta,\kappa},\mathbf{F})$ for each $\kappa\in\Phi_\beta$ according to Theorem 2.5.6 and Corollary 2.5.7 [9]. The net $\{w_{a,\beta}: a\in\Psi_\beta\}$ is norm bounded, since

$$\|B_\beta q_{a,\beta}\|_{L^1(G_\beta)} \le \sup\{\|(B_\beta q_{a,\beta})\tilde{*}f_\alpha\|_{L^1_{G_\beta}(G_\alpha)}:\ f_\alpha\in L^1_{G_\beta}(G_\alpha),\ \|f_\alpha\|_{L^1_{G_\beta}(G_\alpha)}\le 1\}$$

$$= \sup\{\|q_{a,\beta}\tilde{*}(T_\alpha f_\alpha)\|_{L^1_{G_\beta}(G_\alpha)}:\ f_\alpha\in L^1_{G_\beta}(G_\alpha),\ \|f_\alpha\|_{L^1_{G_\beta}(G_\alpha)}\le 1\}\le$$

$$\|T_\alpha\|\sup\{\|q_{a,\beta}\tilde{*}g_\alpha\|_{L^1_{G_\beta}(G_\alpha)}:\ g_\alpha\in L^1_{G_\beta}(G_\alpha),\ \|g_\alpha\|_{L^1_{G_\beta}(G_\alpha)}\le 1\}$$

$$\le \|T_\alpha\|<\infty,\ \text{since}$$

$$\|u_\beta\tilde{*}g_\alpha\|_{L^1_{G_\beta}(G_\alpha)}\le\|u\|_{L^1(G_\beta)}\|g_\alpha\|_{L^1_{G_\beta}(G_\alpha)}$$

for each $u\in L^1(G_\beta)$ and $g_\alpha\in L^1_{G_\beta}(G_\alpha)$ (see Lemma 17.2 [19]). This implies that for each $\epsilon>0$ and $\kappa\in\Phi_\beta$ there exists a compact set $K_{\epsilon,\kappa}$ in $G_{\beta,\kappa}$ such that $w_{a,\beta,\kappa}(G_{\beta,\kappa}\setminus K_{\epsilon,\kappa})<\epsilon$ for each $a\in\Psi_\beta$, since $\mu_{\beta,\kappa}$ as the image of μ_β is the radonian measure on the complete separable metric space $G_{\beta,\kappa}$ and each measure $w_{a,\beta,\kappa}$ is absolutely continuous relative to $\mu_{\beta,\kappa}$ (see also Theorem 1.2 [7] and Formulas $(M1,M2)$).

Applying Theorems either II.24 and II.27 or II.30 [26] we get that a measure $\nu_{\beta,\kappa} \in M_\sigma(G_{\beta,\kappa}, \mathbf{F})$ exists such that the net $w_{a,\beta,\kappa}$ weakly converges to $\nu_{\beta,\kappa}$ for each $\beta \in \Lambda$ and $\kappa \in \Phi_\beta$. Thus using Formulas $(M1, M2)$ we have deduced that

$$(17) \quad \lim_a J_{a,\beta}(f) = \int_{G_\beta} f d\nu_\beta$$

for each $f \in C_b(G_\beta, \mathbf{F})$. The variation of ν_β is finite and $M(G_\beta, \mathbf{F})$ is the Banach space relative to the variation norm according to Theorems I.2 and I.3 [26].

Let $x \in C_b(G_\beta, \mathbf{F})$ and $y \in C_b(G_\gamma, \mathbf{F})$, we consider the function

$$(18) \quad z(g) = \int_{G_\gamma} y(h) x(\theta^\gamma_\beta(h)g)\mu_\gamma(dh).$$

It evidently exists and is μ_β-measurable, since $\mu_\gamma(G_\gamma) = 1$, consequently,

$$\sup_{g\in G_\beta} |\int_{G_\gamma} y(h)x(\theta^\gamma_\beta(h)g)\mu_\gamma(dh)| \le \|y\|_{C_b(G_\gamma,\mathbf{F})}\|x\|_{C_b(G_\beta,\mathbf{F})}.$$

Moreover, $z \in C_b(G_\beta, \mathbf{F}) \subset L^1_{G_\gamma}(G_\beta)$ due to the latter inequality and Properties $(11, 14)$ (see above). Since ν_β is the weak limit of the net $J_{a,\beta}$, then for each $\epsilon > 0$ there exists $b \in \Psi_\beta$ such that

$$(19) \quad |\int_{G_\beta} z(g)\nu_\beta(dg) - \int_{G_\beta} z(g)(B_\beta q_{a,\beta})(g)\mu_\beta(dg)| < \epsilon$$

for each $a > b$. In view of the Fubini theorem the latter inequality implies that

$$(20) \quad |\int_{G_\gamma} y(h)\mu_\gamma(dh)\int_{G_\beta} x(\theta^\gamma_\beta(h)g)\nu_\beta(dg) -$$

$$\int_{G_\gamma} y(h)\mu_\gamma(dh)\int_{G_\beta} x(\theta^\gamma_\beta(h)g)(B_\beta q_{a,\beta})(g)\mu_\beta(dg)| \le \epsilon$$

for each $a > b$. Therefore, $T_\alpha x(g) = (\nu_\beta \tilde{*} x)(g)$ for each $x \in C_b(G_\beta, \mathbf{F}) \cap [(\theta^\beta_\alpha)^{-1}(C_b(G_\alpha, \mathbf{F}))]$ and $g \in G_\beta$. If $f_\alpha \in C_b(G_\alpha, \mathbf{F})$, then its restriction $f_\alpha|_{\theta^\beta_\alpha(G_\beta)}$ is continuous and bounded, that is $f_\alpha \circ (\theta^\beta_\alpha)^{-1}$ is continuous and bounded on (G_β, τ_β) due to 1(2). Moreover, the function $\psi_g(h) := f_\alpha(\theta^\beta_\alpha(h)g)$ is continuous and bounded by $h \in G_\beta$ for each $g \in G_\alpha$. Hence

$$(21) \quad (\nu_\beta \tilde{*} \psi_g)(s) = \int_{G_\beta} f_\alpha(\theta^\beta_\alpha(hs)g)\nu_\beta(dh) = [\nu_\beta \tilde{*} f_\alpha](\theta^\beta_\alpha(s)g)$$

is defined for each $s \in G_\beta$ and $g \in G_\alpha$, particularly for $s = e_\beta$.

By the conditions of this theorem $T_\alpha : L^1_{G_\beta}(G_\alpha) \to L^1_{G_\beta}(G_\alpha)$ is the continuous linear operator. There is also the inclusion $C_b(G_\alpha, \mathbf{F}) \subset L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ so that $C_b(G_\alpha, \mathbf{F})$ is dense in $L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$, since $\mu_\alpha(X_\alpha) = \mu_\alpha(G_\alpha) = 1$ with the σ-compact subset X_α in G_α (see also Lemma 17.8 and Proposition 17.9 [19] and Property (14) above). Let $f_\alpha \in L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ and we take any sequence of bounded continuous functions $f_{\alpha,n} \in C_b(G_\alpha, \mathbf{F})$ converging to f_α in $L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$. We have

$$(22) \quad \lim_a (B_\beta q_{a,\beta}) \tilde{*} f_{\alpha,n} = f_\alpha \text{ and } \lim_n f_{\alpha,n} = f_\alpha$$

in $L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$. Then

$$(23) \quad \|(B_\beta q_{a,\beta}) \tilde{*} f_{\alpha,n} - (B_\beta q_{b,\beta}) \tilde{*} f_{\alpha,m}\|_{L^1_{G_\beta}(G_\alpha)} \le$$

$$\|(B_\beta q_{a,\beta} - B_\beta q_{b,\beta}) \tilde{*} f_{\alpha,n}\|_{L^1_{G_\beta}(G_\alpha)} + \|(B_\beta q_{b,\beta}) \tilde{*}\| \|f_{\alpha,n} - f_{\alpha,m}\|_{L^1_{G_\beta}(G_\alpha)},$$

consequently, for each $\epsilon > 0$ there exist $a_0 \in \Psi_\beta$ and $n_0 \in \mathbf{N}$ such that

$$(24) \quad \|(B_\beta q_{a,\beta}) \tilde{*} f_{\alpha,n} - (B_\beta q_{b,\beta}) \tilde{*} f_{\alpha,m}\|_{L^1_{G_\beta}(G_\alpha)} < \epsilon$$

for each $a, b > a_0$ and $n, m > n_0$ (see Lemma 17.2 and Proposition 17.7 [19] and Formula $(B1)$ above). That is the net $\{(B_\beta q_{a,\beta}) \tilde{*} f_{\alpha,n} : (a,n)\}$ is fundamental (i.e. of the Cauchy type) in the Banach space $L^1_{G_\beta}(G_\alpha)$, where $(a,n) \le (b,m)$ if $a \le b$ and $n \le m$. Therefore the limit exists

$$(25) \quad T_\alpha f_\alpha = \lim_{a,n} (B_\beta q_{a,\beta}) \tilde{*} f_{\alpha,n} = \lim_n \lim_a (B_\beta q_{a,\beta}) \tilde{*} f_{\alpha,n} = \lim_n \nu_\beta \tilde{*} f_{\alpha,n} = \nu_\beta \tilde{*} f_\alpha.$$

Thus

$$T_\alpha f_\alpha = \nu_\beta \tilde{*} f_\alpha$$

for each $f_\alpha \in L^1_{G_\beta}(G_\alpha)$ as well, that is, Formulas $(1,2)$ are fulfilled.

9. Theorem. *Let suppositions of Theorem 8 be satisfies. Then the statement of Theorem 8 is equivalent to the following:*

(1) *relative to the strong operator topology the set of all convolution operators of the form 8(1,2) on* $\mathcal{E} := L^\infty(L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F}) : \alpha < \beta \in \Lambda)$ *with values in* $\mathcal{E}$ *is a closed subset of the ring of all bounded linear operators from* $\mathcal{E}$ *into* $\mathcal{E}$.

Proof. $(8 \Rightarrow 9)$. Let $\nu_{a,\beta}\tilde{*}$ be a net of convolution operators converging to an operator $T_\alpha : L^1_{G_\beta}(G_\alpha) \to L^1_{G_\beta}(G_\alpha)$ in the strong operator topology for each $\alpha \in \Lambda$, hence T is the left meta-centralizer on $\mathcal{E}$, since each operator $\{\nu_{a,\beta}\tilde{*} : \alpha \in \Lambda, \beta = \phi(\alpha)\}$ is the left meta-centralizer.

$(9 \Rightarrow 8)$. From the proof of Theorem 8 we analogously get

$$T_\alpha f_\alpha = \lim_a \nu_{a,\beta}\tilde{*} f_\alpha$$

for each $\alpha \in \Lambda$ and $f_\alpha \in L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ with $\beta = \phi(\alpha)$, where $\nu_{a,\beta} \in M(G_\beta, \mathbf{F})$ for each $\beta \in \Lambda$ and $a \in \Psi_\beta$ consequently, $T = (T_\alpha : \alpha)$ is the convolution operator.

10. Theorem. *Let S be a bounded linear mapping of $\mathcal{E}$ (see §4) into itself such that $Sf = (S_\alpha f_\alpha : \alpha \in \Lambda)$ with $S_\alpha : L^1_{G_\beta}(G_\alpha) \to L^1_{G_\beta}(G_\alpha)$ for each $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$. Then the following statements (i) and (ii) are equivalent:*

(i) *an operator S has the form*

(1) *$S = p\hat{U}_a$ for some marked elements $a \in G := \prod_{\alpha\in\Lambda} G_\alpha$ and $p = \{p_\alpha : |p_\alpha| = 1 \ \forall \alpha \in \Lambda\} \in \mathbf{F}^\Lambda$, that is*

(2) *$S_\alpha f_\alpha(x) = p_\alpha \hat{U}_{a_\beta} f_\alpha(x)$ for any $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$ and each $x \in G_\alpha$, where*

(3) *$\hat{U}_{g_\beta} f_\alpha(x) = f_\alpha(\theta^\beta_\alpha(g_\beta)x)$ for each $g_\beta \in G_\beta$ and $x \in G_\alpha$;*

(ii) *(4) S is a left meta-centralizer and*

(5) *$\|S_\alpha f_\alpha\| = \|f_\alpha\|$ for every $f_\alpha \in L^1_{G_\beta}(G_\alpha)$ and $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$.*

Proof. The $\mathbf{F}$-linear span of the set of all non-negative functions $f \in L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ is dense in $L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$. Therefore, each bounded linear operator S_α can be written in the form $S_\alpha = S_{1,\alpha} + iS_{2,\alpha} = S^+_{1,\alpha} - S^-_{1,\alpha} + iS^+_{2,\alpha} - iS^-_{2,\alpha}$, where $S^+_{k,\alpha} f \geq 0$ and $S^-_{k,\alpha} f \geq 0$ for $k = 1, 2$ and each $f \in P_\alpha$, $S_{k,\alpha} = S^+_{k,\alpha} - S^-_{k,\alpha}$, where P_α denotes the cone of functions in $L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ non-negative μ_α-almost everywhere on G_α. Certainly over the real field additives $S^\pm_{2,\alpha}$ vanish. In view of Theorem 11 [17] there exist $a^+_k \in G$ and $p^+_k = \{p^+_{k,\alpha} : p^+_{k,\alpha} > 0 \ \forall \alpha \in \Lambda\} \in \mathbf{R}^\Lambda$ such that $S^+_{k,\alpha} f_\alpha(x) = p^+_{k,\alpha} \hat{U}_{a^+_{k,\beta}} f_\alpha(x)$ and analogously for $S^-_{k,\alpha}$ for each $k = 1, 2$.

Suppose that $a^t_k \neq a^s_l$ for some $t, s \in \{+, -\}$ and $k, l \in \{1, 2\}$, then there exists $\alpha \in \Lambda$ such that $a^t_{k,\beta} \neq a^s_{l,\beta}$ with $\beta = \phi(\alpha)$. On the other hand, we have $S_{k,\alpha} f_\alpha = S^+_{k,\alpha} f_\alpha - S^-_{k,\alpha} f_\alpha = p^+_{k,\alpha} f_\alpha(\theta^\beta_\alpha(a^+_{k,\beta})x) - p^-_{k,\alpha} f_\alpha(\theta^\beta_\alpha(a^-_{k,\beta})x)$ for each $f_\alpha \in L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$, since $f_\alpha = [f^+_{1,\alpha} - f^-_{1\alpha}] + i[f^+_{2,\alpha} - f^-_{2,\alpha}]$, where

$f^+_{k,\alpha}(x) = \max(f_{k,\alpha}(x),0)$ for every $k = 1,2$ and $x \in G_\alpha$, $f^+_{k,\alpha}, f^-_{k,\alpha} \in P_\alpha$. Then if U is an open subset in G_α such that $\theta^\beta_\alpha(a^s_{k,\beta})U \cap \theta^\beta_\alpha(a^t_{l,\beta})U = \emptyset$ for every $k,l = 1,2$ and $t,s \in \{+,-\}$, then $\|S_\alpha \chi_U\| = \sum_{k=1}^2 \sum_{t\in\{+,-\}}(|p^t_{k,\alpha}|\|\hat{U}_{a^t_{k,\beta}}\chi_U\|)$. If the interior of the intersection $\cap_{k=1}^2 \cap_{t\in\{+,-\}} (\theta^\beta_\alpha(a^t_{k,\beta})U)$ is non-void, then $\|S_\alpha\chi_U\| < \sum_{k=1}^2 \sum_{t\in\{+,-\}}(|p^t_{k,\alpha}|\|\hat{U}_{a^t_{k,\beta}}\chi_U\|)$, since $\mu_\alpha(V) > 0$ for each open subset V in G_α, consequently, S_α is not an isometry.

Therefore, if S satisfies Conditions $ii(4,5)$, then $a^t_{k,\beta} = a^s_{l,\beta}$ for each $t,s \in \{+,-\}$ and $k,l \in \{1,2\}$. Thus $(S_\alpha f_\alpha) = p_\alpha \hat{U}_{a_\beta} f_\alpha(x)$ for any $\alpha \in \Lambda$ and each $x \in G_\alpha$, where $p_\alpha = p^+_{1,\alpha} - p^-_{1,\alpha} + ip^+_{2,\alpha} - ip^-_{2,\alpha}$. Naturally, in the case $\mathbf{F} = \mathbf{R}$ the terms $p^\pm_2$ vanish. In view of Lemma 7 [17] $\hat{U}_a$ is the isometry. Since S preserves norms, then $|p_\alpha| = 1$ for each α.

Vice versa Conditions $i(1-3)$ imply $ii(4,5)$ due to Lemma 7 [17].

11. Lemma. *Let $\hat{U}_c$ be a left translation on $\mathcal{E}$ as in §10, let also $T : \mathcal{E} \to \mathcal{F}$ be an isomorphism of normed algebras such that $Tf = (T_\alpha f_\alpha : \alpha \in \Lambda)$, $T_\alpha : L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F}) \to L^1_{H_\beta}(H_\alpha, \lambda_\alpha, \mathbf{F})$ and $\|T_\alpha\| \le 1$ for each α, where $\mathcal{F} = L^\infty(L^1_{H_\beta}(H_\alpha, \lambda_\alpha, \mathbf{F}) : \alpha < \beta \in \Lambda)$. If $\hat{K}_c = T\hat{U}_c T^{-1}$, then there exist mappings of groups $\xi : G \to H$ and $p : G \to \mathbf{F}^\Lambda$ such that*

(1) *$\hat{K}_c = p_c \hat{V}_t$ for $t = \xi(c)$ and*

(2) *$p_c = \{p_{c,\alpha} : |p_{c,\alpha}| = 1\ \forall \alpha \in \Lambda\} \in \mathbf{F}^\Lambda$, where $\hat{V}_d$ denotes the left translation operator on $\mathcal{F}$, $c \in G$.*

Proof. We have $T(f\tilde{\star}u) = (Tf)\tilde{\star}(Tu)$ for each $u,f \in \mathcal{E}$ and $T^{-1}(g\tilde{\star}v) = (T^{-1}g)\tilde{\star}(T^{-1}v)$ for each $v,g \in \mathcal{F}$. One can take the approximate identity $\{q_{a,\beta} : a \in \Psi_\beta\}$ as in §8 and consider functions $s_{a,\beta} = T_\beta q_{a,\beta}$. The operator T is bijective and continuous from $\mathcal{E}$ onto $\mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ as linear normed spaces are complete. According to the Banach theorem IV.5.4.3 [15] (or see [1]) the inverse operator T^{-1} is also bounded. Due to Formulas 8(7,8) there exists the adjoint operator $(\hat{K}_{c_\gamma})^*$ relative to the $\tilde{*}$ multiplication for each $c \in G$ and $\gamma \in \Lambda$. For each $f,g \in \mathcal{F}$, $\gamma = \phi(\beta)$ and $\beta = \phi(\alpha)$ the limit exists

$$(\hat{K}_{c_\gamma} f_\beta)\tilde{*}g_\alpha = f_\beta\tilde{*}[(\hat{K}_{c_\gamma})^* g_\alpha] = \lim_a f_\beta\tilde{*}\{s_{a,\beta}\tilde{*}[(\hat{K}_{c_\gamma})^* g_\alpha]\}$$

$$= f_\beta\tilde{*}\{\lim_a(\hat{K}_{c_\gamma} s_{a,\beta})\tilde{*}g_\alpha\} = f_\beta\tilde{*}\{\lim_a(T_\beta\hat{U}_{c_\gamma}T_\beta^{-1}T_\beta q_{a,\beta})\tilde{*}g_\alpha\}$$

$$= f_\beta\tilde{*}\{\lim_a(T_\beta\hat{U}_{c_\gamma}q_{a,\beta})\tilde{*}g_\alpha\} \text{ and hence}$$

$$\|(\hat{K}_{c_\gamma} f_\beta)\tilde{*}g_\alpha\| \le$$

$$\overline{\lim}_a \|f_\beta \tilde{*}([T_\beta \hat{U}_{c_\gamma} q_{a,\beta}]\tilde{*} g_\alpha)\| \le \|f_\beta\| \|T_\beta\| \|g_\alpha\| \ \overline{\lim}_a \|[\hat{U}_{c_\gamma} q_{a,\beta}]\tilde{*}\| \le \|f_\beta\| \|g_\alpha\|$$

for each $f, g \in \mathcal{E}$, since $\|T\| \le 1$. On the other hand, $\hat{K}_{c_\gamma^{-1}} = (\hat{K}_{c_\gamma})^{-1}$. Thus the inequalities $\|\hat{K}_{c_\gamma}\| \le 1$ and $\|(\hat{K}_{c_\gamma})^{-1}\| \le 1$ are satisfied for each $\gamma \in \Lambda$ and $c \in G$, consequently, $\hat{K}_c$ is the isometry for each $c \in G$.

Applying Theorem 10 we get the statement of this lemma.

12. Lemma. *The mappings $(G, \tau_G^b) \ni c \to p_c \in (B^\Lambda, \tau_B^b)$ for each β and $(G, \tau_G^b) \ni c \mapsto \xi(c) \in (H, \tau_H^b)$ of Lemma 11 are continuous homomorphisms, where $B = \{x \in \mathbf{F} : \ |x| = 1\}$ is the multiplicative group, the product B^Λ is in the box topology τ_B^b, where τ_G^b denotes the box topology on G (see §9 [17]).*

Proof. These mappings are homomorphisms, since

$$p_{ch,\gamma} \hat{V}_{\xi_\gamma(c_\gamma h_\gamma)} = T_\beta \hat{U}_{c_\gamma h_\gamma} T_\beta^{-1} = T_\beta \hat{U}_{c_\gamma} T_\beta^{-1} T_\beta \hat{U}_{h_\gamma} T_\beta^{-1} = p_{c,\gamma} \hat{V}_{\xi_\gamma(c_\gamma)} p_{h,\gamma} \hat{V}_{\xi_\gamma(h_\gamma)}$$

for each $c, h \in G$, $\beta \in \Lambda$ with $\gamma = \phi(\beta)$, where $\xi(c) = \{\xi_\alpha(c_\alpha) : \ \alpha \in \Lambda\}$, $\xi_\alpha : G_\alpha \to H_\alpha$ for each $\alpha \in \Lambda$. The mapping ξ is bijective, since for $\xi(c) = e_H \in H$, where e_H is the neutral element in H, one gets $p_{c,\gamma} I_{\mathcal{F}} = T_\beta \hat{U}_{c_\gamma} T_\beta^{-1}$ and hence $\hat{U}_{c_\gamma} = p_{c,\gamma} I_{\mathcal{E}}$, where $I_{\mathcal{E}}$ denotes the unit operator on $\mathcal{E}$. Therefore, $c = e_G$ and hence $p_{c,\gamma} = 1$ for each γ.

Then the mapping $G \ni c \mapsto \hat{U}_c$ is continuous from G in the box topology τ_G^b and relative to the strong operator topology according to Proposition 10 [17], consequently, the mapping $H \ni t \mapsto \hat{V}_t$ is also continuous, since T and T^{-1} are bounded linear operators.

Then for each $\epsilon = (\epsilon_\alpha > 0 : \alpha \in \Lambda)$ there exists a neighborhood $Y = \prod_{\alpha \in \Lambda} Y_\alpha$ of e_H in (H, τ_H^b) such that each Y_α is an (open) neighborhood of the neutral element e_α in H_α for which $\epsilon_\alpha/2 < \lambda_\alpha(Y_\alpha) < \epsilon_\alpha$ for each $\alpha \in \Lambda$, since λ_α is the quasi-invariant borelian measure on H_α relative to the dense subgroup H_β and hence non-atomic. Moreover, if Z is an arbitrary neighborhood of e_H in (H, τ_H^b), then there exists Y such that $YY^{-1} \subseteq Z$. Then the function $g = (g_\alpha = \chi_{Y_\alpha} : \alpha \in \Lambda)$ belongs to $\mathcal{F}$, where χ_{A_α} denotes the characteristic function of a subset A_α in H_α. Suppose that p is a marked element in B^Λ. Let $t \in H$ be such that

$$(1) \quad \|p_\beta g_\beta \tilde{*}(\hat{V}_{t_\beta}^* g_\alpha) - g_\beta \tilde{*} g_\alpha\| < [\lambda_\beta|_{Y_\beta} \tilde{*} \lambda_\alpha](Y_\alpha), \text{ where}$$

$$[\lambda_\beta|_{Y_\beta} \tilde{*} \lambda_\alpha](Y_\alpha) := \int_{Y_\beta} \int_{Y_\alpha} \lambda_\beta(dx_\beta) \lambda_\alpha(\theta_\alpha^\beta(x_\beta) dx_\alpha),$$

where $\theta^\beta_\alpha : H_\beta \hookrightarrow H_\alpha$ are embeddings (see §1). If $t_\beta \notin Z_\beta$, then $s_\beta Y_\beta$ and $s_\beta t_\beta Y_\beta$ are the disjoint subsets in the group H_β for each element s_β in H_β, consequently,

$$\|p_\beta g_\beta \tilde{*} [\hat{V}^*_{t_\beta} g_\alpha] - g_\beta \tilde{*} g_\alpha\| = \sup_{s_\beta \in H_\beta} \int_{H_\alpha} |p_\beta [\hat{V}_{s_\beta t_\beta} g_\beta] \tilde{*} g_\alpha(x_\alpha) - [\hat{V}_{s_\beta} g_\beta] \tilde{*} g_\alpha(x_\alpha)| \lambda_\alpha(dx_\alpha)$$

$$= \sup_{s_\beta \in H_\beta} \int_{H_\beta} \int_{H_\alpha} |p_\beta g_\beta(s_\beta t_\beta x_\beta) g_\alpha(\theta^\beta_\alpha(x_\beta) x_\alpha)| \lambda_\beta(dx_\beta) \lambda_\alpha(dx_\alpha)$$

$$+ \sup_{s_\beta \in H_\beta} \int_{H_\beta} \int_{H_\alpha} |g_\beta(s_\beta x_\beta) g_\alpha(\theta^\beta_\alpha(x_\beta) x_\alpha)| \lambda_\beta(dx_\beta) \lambda_\alpha(dx_\alpha) \ge [\lambda_\beta|_{Y_\beta} \tilde{*} \lambda_\alpha](Y_\alpha).$$

Thus Inequality (1) implies that $t_\beta \in Z_\beta$. Hence the mapping $p\hat{V}_{\xi_\beta(c_\beta)} \mapsto \xi_\beta(c_\beta) = t_\beta \in H_\beta$, with H_β in the topology τ_β, is continuous for each β, when linear operators $p\hat{V}$ are considered relative to the strong operator topology, since the set of all (μ_α-measurable) simple functions is dense in $L^1_{G_\beta}(G_\alpha)$. The mapping $c_\beta \mapsto \xi_\beta(c_\beta)$ is the composition of three mappings $c_\beta \mapsto \hat{U}_{c_\beta} \mapsto T_\alpha \hat{U}_{c_\beta} T_\alpha^{-1} = p_{c,\beta} \hat{V}_{\xi_\beta(c_\beta)} \mapsto \xi_\beta(c_\beta) = t_\beta$ which are continuous for each $\beta \in \Lambda$ as it was proved above, consequently, the mapping $\xi : (G, \tau^b_G) \to (H, \tau^b_H)$ is also continuous.

The mapping $c \mapsto p_c$ is continuous, since $c \mapsto p_c I$ is continuous as the composition of two uniformly bounded and continuous mappings $T\hat{U}_c T^{-1}$ and $\hat{K}_{\xi(c)}$.

13. Lemma. *The mapping $\xi : G \to H$ is the homeomorphism of (G, τ^b_G) onto (H, τ^b_H).*

Proof. If $\{\xi_\beta(x_{\beta,b}) : b\}$ is a net converging to $y_\beta \in H_\beta$, where $x_{\beta,b} \in G_\beta$, then $\{\hat{V}_{\xi_\beta(x_{\beta,b})} : b\}$ converges to $\hat{V}_{y_\beta}$ in the strong operator topology. Therefore, $\{T_\alpha^{-1} \hat{V}_{\xi_\beta(x_{\beta,b})} T_\alpha : b\}$ converges to $T_\alpha^{-1} \hat{V}_{y_{\beta,b}} T_\alpha$. From Lemma 11 we have the equality

$$T_\alpha^{-1} \hat{V}_{\xi_\beta(x_{\beta,b})} T_\alpha = p^{-1}_{x_b,\beta} \hat{U}_{x_{\beta,b}},$$

hence the net of operators $\{p^{-1}_{x_b,\beta} \hat{U}_{x_{\beta,b}} : b\}$ strongly converges to $p_\beta \hat{U}_{x_\beta}$ for some $p_\beta \in B$ and $x_\beta \in G_\beta$. Thus the equality

$$p_\beta T_\alpha \hat{U}_{x_\beta} T_\alpha^{-1} = \hat{V}_{y_\beta}$$

is fulfilled with $y_\beta = \xi_\beta(x_\beta)$ and $p_\beta = p^{-1}_{x,\beta}$ for each $\beta \in \Lambda$. This implies that $\xi_\beta(G_\beta)$ is closed in H_β for each β and hence $\xi(G)$ is closed in (H, τ^b_H).

The inverse operator T^{-1} is bounded (see §11). Then $T_\alpha^{-1}\hat{V}_{y_\beta}T_\alpha = (sT_\alpha)^{-1}\hat{V}_{y_\beta}(sT_\alpha)$ for each $s \in \mathbf{F} \setminus \{0\}$. Hence without loss of generality we can consider that $0 < \|T_\alpha^{-1}\| \le 1$ for each $\alpha \in \Lambda$. On the other hand, from the equality $T_\alpha^{-1}\hat{V}_{y_\beta}T_\alpha = p_{x,\beta}^{-1}\hat{U}_{x_\beta}$ with $x_\beta = \xi_\beta^{-1}(y_\beta)$ analogously to ξ in §12 the continuity of $\xi_\beta^{-1} : \xi_\beta(G_\beta) \to G_\beta$ follows.

Applying Lemmas 11, 12 and the proof in this section above to $T^{-1} : \mathcal{F} \to \mathcal{E}$ we get that there exists a continuous bijective homomorphism $\eta : (H, \tau_H^b) \to (G, \tau_G^b)$ such that $\eta(H)$ is closed in (G, τ_G^b) and

(1) $\hat{Q}_y = r_y \hat{U}_t$ for $t = \eta(y)$ and

(2) $r_y = \{r_{y,\alpha} : |r_{y,\alpha}| = 1 \ \forall \alpha \in \Lambda\} \in \mathbf{F}^\Lambda$, where $\hat{Q}_y = T^{-1}\hat{V}_y T$ for each $y \in H$, $r : (G, \tau_G^b) \to B^\Lambda$ is a continuous homomorphism. The operators $\hat{K}_c$ and $\hat{Q}_y$ are the left meta-centralizers on $\mathcal{F}$ and $\mathcal{E}$ respectively for each $c \in G$ and $y \in H$. But from 11(1, 2) it follows that $\eta = \xi^{-1}$ and $p_{\eta(y)} = r_y^{-1}$ for each $y \in H$, since η and ξ are bijective homomorphisms. Therefore, Formulas $(1, 2)$ and $11(1, 2)$ imply that $\eta(\xi(G)) = G$ and hence $\xi(G) = H$.

14. Theorem. *Let $T : \mathcal{E} \to \mathcal{F}$ be an isomorphism of normed algebras such that $Tf = (T_\alpha f_\alpha : \alpha \in \Lambda)$, $T_\alpha : L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F}) \to L^1_{H_\beta}(H_\alpha, \lambda_\alpha, \mathbf{F})$ and $\|T_\alpha\| \le 1$ for each α, where $\mathcal{F} = L^\infty(L^1_{H_\beta}(H_\alpha, \lambda_\alpha, \mathbf{F}) : \alpha < \beta \in \Lambda)$ (see §§11, 12). Then a homeomorphism ξ of topological groups exists from (G, τ_G^b) onto (H, τ_H^b) and a continuous homomorphism $\psi : G \to B^\Lambda$ such that*

(1) $T\hat{U}_x T^{-1} = \psi(x^{-1})\hat{V}_{\xi(x)}$ and

(2) $(Tf)_\alpha(\xi(x)) = \psi_\beta(x_\beta) f_\alpha(x_\alpha)$ for each $x \in G$, $f \in \mathcal{E}$ and $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$, where $\psi(x) = (\psi_\alpha(x_\alpha) : \alpha \in \Lambda)$, $\psi_\alpha : G_\alpha \to B$,

$$T_\alpha \hat{U}_{x_\beta} T_\alpha^{-1} = \psi_\beta(x_\beta^{-1}) \hat{V}_{\xi_\beta(x_\beta)}.$$

Moreover, T is an isometry.

Proof. We define a homomorphism $\psi(x) = p_x^{-1}$, hence $\psi(x) = (\psi_\alpha(x_\alpha) = p_{x,\alpha}^{-1} : \alpha \in \Lambda\} \in B^\Lambda$, hence $\psi_\alpha : G_\alpha \to B$ is a character for each $\alpha \in \Lambda$. From Lemmas 11-13 Statement (1) of this theorem follows such that $\xi : (G, \tau_G^b) \to (H, \tau_H^b)$ and $\xi^{-1} : (H, \tau_H^b) \to (G, \tau_G^b)$ and $\psi : G \to B^\Lambda$ are continuous homomorphisms with $\xi(G) = H$.

If $S : \mathcal{E} \to \mathcal{F}$ is an isomorphism of normed algebras such that $Sf = (S_\alpha f_\alpha : \alpha \in \Lambda)$, $S_\alpha : L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F}) \to L^1_{H_\beta}(H_\alpha, \lambda_\alpha, \mathbf{F})$ and $\|S_\alpha\| \le 1$ for each α such that S satisfies Equality (2):

$(Sf)_\alpha(\xi(x)) = \psi_\beta(x_\beta) f_\alpha(x_\alpha)$ for each $x \in G$ and $f \in \mathcal{E}$, then $(S^{-1}g)_\alpha(x) =$

$\psi_\beta(x_\beta^{-1})g_\alpha(\xi_\alpha(x_\alpha))$ for each $g\in\mathcal{F}$ and $x\in G$. Therefore one infers that

$$(S_\alpha\hat{U}_{c_\beta}S_\alpha^{-1}g_\alpha)(\xi_\alpha(x_\alpha))=\psi_\beta(x_\beta)(\hat{U}_{c_\beta}S_\alpha^{-1}g_\alpha)(x_\alpha)=$$

$$\psi_\beta(x_\beta)(S_\alpha^{-1}g_\alpha)(\theta_\alpha^\beta(c_\beta)x_\alpha)=\psi_\beta(x_\beta)\psi_\beta(x_\beta^{-1}c_\beta^{-1})g_\alpha(\theta_\alpha^\beta(\xi_\beta(c_\beta))\xi_\alpha(x_\alpha))$$
$$=\psi_\beta(c_\beta^{-1})g_\alpha(\theta_\alpha^\beta(\xi_\beta(c_\beta))\xi_\alpha(x_\alpha))=\psi_\beta(c_\beta^{-1})(\hat{U}_{\xi_\beta(c_\beta)}g_\alpha)(\xi_\alpha(x_\alpha)),$$

consequently, $S_\alpha\hat{U}_{c_\beta}S_\alpha^{-1}=\psi_\beta(c_\beta^{-1})\hat{U}_{\xi_\beta(c_\beta)}$ for each $c\in G$, $\alpha\in\Lambda$ with $\beta=\phi(\alpha)$, where embeddings $H_\beta\hookrightarrow H_\alpha$ also are denoted by θ_α^β for the notation simplicity (see §1). This means that $S\hat{U}_cS^{-1}=T\hat{U}_cT^{-1}$ and hence

$$(3)\quad T_\alpha^{-1}S_\alpha\hat{U}_{c_\beta}=\hat{U}_{c_\beta}T_\alpha^{-1}S_\alpha$$

for each $\alpha\in\Lambda$ with $\beta=\phi(\alpha)$. In view of Lemmas 11-13 and the conditions of this theorem the linear operators T, T^{-1}, S and S^{-1} are continuous. Thus the operator

(4) $T^{-1}S=:Y$

is the isomorphism of the algebra $\mathcal{E}$ onto itself commuting with all operators $\hat{U}_c$ such that Y and Y^{-1} are continuous. As in §13 it is sufficient to consider the case $0<\|Y_\alpha\|\le 1$ for each $\alpha\in\Lambda$, since $\hat{U}_{c_\beta}=Y_\alpha^{-1}\hat{U}_{c_\beta}Y_\alpha=(kY_\alpha)^{-1}\hat{U}_{c_\beta}(kY_\alpha)$ for every $k\in\mathbf{F}\setminus\{0\}$, $\alpha\in\Lambda$ with $\beta=\phi(\alpha)$ and $c\in G$. Take $f,q\in\mathcal{E}$ and consider the left meta-centralizer A defined by a radonian measure $\nu_\alpha\in M_t(G_\alpha,\mathbf{F})$ such that

(5) $\nu_\alpha(dx_\alpha)=q_\alpha(x_\alpha)\mu_\alpha(dx_\alpha)$

for each $\alpha\in\Lambda$, that is $Af=\nu\tilde{\star}f$. On the other hand,

$$(6)\quad (Af)_\alpha(x_\alpha)=\int_{G_\beta}q_\beta(y_\beta)[\hat{U}_{y_\beta}f_\alpha(x_\alpha)]\mu_\beta(dy_\beta),$$

that is relative to the strong operator topology

$$(7)\quad A_\alpha=\int_{G_\beta}q_\beta(y_\beta)\hat{U}_{y_\beta}\mu_\beta(dy_\beta)$$

for each $\alpha\in\Lambda$ with $\beta=\phi(\alpha)$, where $Af=(A_\alpha f_\alpha:\alpha\in\Lambda)$. In each Banach space $L^1_{G_\gamma}(G_\beta,\mu_\beta,\mathbf{F})$ the space of (μ_β-measurable) simple functions $\sum_{j=1}^n v_j\chi_{Z_j}$ is dense, where $v_j\in\mathbf{F}$ is a constant and Z_j is a μ_β-measurable

subset in G_β for each $j = 1, .., n$, $n \in \mathbf{N}$. Therefore, from Formulas $(3-7)$ it follows that

$$YAf = Y(q\tilde{\star}f) = (Yq)\tilde{\star}(Yf) = AYf = q\tilde{\star}(Yf),$$

consequently, $Yq = q$ for each $q \in \mathcal{E}$, since $f \in \mathcal{E}$ is arbitrary. Thus $Y = I_{\mathcal{E}}$ and hence $T = S$, where $I_{\mathcal{E}}$ denotes the unit operator on $\mathcal{E}$. From this Formula (2) follows. The last statement follows from Formulas (2) and 3(1).

15. Remark. The results of this paper can be used for further studies of non locally compact group algebras, representations of groups, completions and extensions of groups, etc.

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